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Examples of quantisation of Poisson manifolds

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Abstract

In quantum physics, the operators associated with the position and the momentum of a article are unbounded operators and C^*-algebraic quantisation does therefore not deal with such operators. In the present article, I propose a quantisation of the Lie–Poisson structure of the dual of a Lie algebroid which deals with a big enough class of functions to include the above-mentioned example. As an application, I show with an example how the quantisation of the dual of the Lie algebroid associated to a Poisson manifold can lead to a quantisation of the Poisson manifold itself. The example, I consider is the torus with constant Poisson structure, in which case I recover its usual C^*-algebraic quantisation.

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1. Introduction

In his PhD thesis, Ramazan [5] (see also Landsman and Ramazan [2]) proved a conjecture of Landsman which roughly speaking states that the quantised, that is deformed, algebra of functions on the dual of a Lie algebroid in the direction of its natural Lie-Poisson bracket is the C^*-algebra of the Lie groupoid integrating the Lie algebroid.[1] The type of quantisations that Ramazan considers are deformation quantisations in the sense of Rieffel [6]. Not all functions are quantised in this way, in fact only functions whose Fourier transform is compactly supported (with respect to a given family of measures) are quantised.

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[1] That is if the Lie algebroid is integrable.

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If M is a Riemannian manifold, then its tangent bundle is a Lie algebroid which integrates to the pair groupoid $M \times M$. The induced Lie–Poisson structure on T^*M is the usual symplectic structure on a cotangent bundle and Landsman–Ramazan's quantisation can be carried over. Nevertheless, this example shows an important limitation of this procedure: functions that are polynomials in the fibres of $T^*M \to M$ are not quantised, whereas functions giving the position or the momentum of a particle are of this type. Moreover, it is well known by physicists that the quantisation of such functions are unbounded operators, whereas Landsman–Ramazan's quantisation only gives elements of C^*-algebras, that is bounded operators on a Hilbert space.

In the present article, I wish to propose a quantisation of the dual of an integrable Lie algebroids $A \to M$ which can be used on a wide class of functions. This class contains in particular functions which are polynomial in the fibres of $A^* \to M$. This will be done in Sections 2 and 3, where Theorem 2.14 is the main result. In Section 4, I show that Theorem 2.14 can be used to recover the physicists' position and momentum operators of a particle moving in $\mathbb{R}^n$.

If M is a Poisson manifold, then its cotangent bundle is naturally a Lie algebroid whose dual can be quantised using Theorem 2.14. One might then hope that this quantisation will help finding a quantisation of the original Poisson manifold M. This slightly naive idea is shown to work on an example, the torus with constant Poisson structure, in Sections 5 and 6. There, I recover the usual C^*-algebraic quantisation of a constant Poisson structure on a torus (see Tang and Weinstein [7], and Weinstein [8]). Part of the strategy of Section 6 consists in finding a Poisson map between TM and M. Such maps are solutions to a partial differential equation derived in Section 5. In Appendix A, I show how to find a solution to this equation in the case of the sphere in $\mathbb{R}^3$.

2. Method of quantisation

Let $G \rightrightarrows M$ be a groupoid and $\tau : A \to M$ its Lie algebroid. I will use the same letter τ to denote the projection $A^* \to M$ of the dual of A. Choose a Riemannian metric on $A \to M$. By duality, this also gives a Riemannian metric on A^*. I will denote by X, Y or Z elements in A^* and by ξ or ζ elements in A.

Definition 2.1. Let E be a s-family of operators on G, that is a map $q \mapsto E_q$ from M to the linear forms on $C_c^\infty(s^{-1}(q))$. I will denote $C_c^\infty(G) \otimes_s C_c^\infty(M)$ the vector space of such operators which in addition satisfy: for all smooth family of functions H on G with compact support, that is for all compactly supported smooth function H on $N \times G$ for some manifold N, the function:

$$N \times M \to C, \qquad (u, q) \mapsto E_q(x \mapsto H(u, x))$$

is smooth and compactly supported.

Also I will denote $\mathfrak{D}p(G)$ the vector space of s-family of operators E which in addition satisfy: for all smooth family of functions H on G with compact support, that is for all compactly supported smooth function H on $N \times G$ for some manifold N, the function

$$N \times G \to C, \qquad (u, z) \mapsto E_{t(z)}(x \mapsto H(u, xz))$$

is smooth and compactly supported.

Notice that since M is a closed sub-manifold of G, the space $\mathfrak{D}p(G)$ is included in $C_c^\infty(G) \otimes_s C_c^\infty(M)$. On the contrary,

Proposition 2.2. *Let D be in $C_c^\infty(G) \otimes_s C_c^\infty(M)$. For any compactly supported smooth function H on $N \times M$, the map*:

$$N \times G \to C, \qquad (u, z) \mapsto D_{t(z)}(H(u, \cdot z))$$

is smooth. Nevertheless, it might fail to be compactly supported.

Proof. Let H be a compactly supported smooth function on $N \times G$, where N is a manifold. I wish to prove that the map:

$$N \times G \to C, \qquad (u, z) \mapsto D_{t(z)}(H(u, \cdot z))$$

is smooth. Let (u_0, z_0) be a point in $N \times G$. Let φ be a compactly supported smooth function on G such that $\varphi \equiv 1$ on a neighbourhood of z_0. Consider:

$$\tilde{H} : N \times G^{(2)} \to C, \qquad (u, x, z) \mapsto H(u, x \cdot z)\varphi(z).$$

If

$$K' = \{(u, x, z) \in N \times G^{(2)} | (u, xz) \in \operatorname{supp} H, \quad z \in \operatorname{supp} \varphi\},$$

then the support of $\tilde{H}$ is a closed subset of K' and since one easily checks that K' is compact, it follows that $\tilde{H}$ has compact support. Using the fact that $G^{(2)}$ is a closed sub-manifold of $G \times G$, I extend $\tilde{H}$ to a function, still denoted $\tilde{H}$, in $C_c^\infty(N \times G \times G)$. Let $\tilde{N} = N \times G$. By interpreting the extended version of $\tilde{H}$ as a function:

$$\tilde{N} \times G \to C, \qquad ((u, z), x) \mapsto \tilde{H}(u, x, z).$$

I can apply D and obtain a function:

$$\tilde{N} \times M \to C, \qquad ((u, z), q) \mapsto D_q(\tilde{H}(u, \cdot, z))$$

in $C_c^\infty(\tilde{N} \times M)$. The closed sub-manifold $\{x, z, t(z)\}$ of $\tilde{N} \times G$ is diffeomorphic to $N \times G$. Therefore, $(u, z) \mapsto D_{t(z)}(H(u, \cdot z)\varphi(z))$ is smooth. This map is equal to $(u, z) \mapsto D_{t(z)}(H(u, \cdot z))$ in a neighbourhood of (u_0, z_0). This can be done for any choice of (u_0, z_0), it follows that $(u, z) \mapsto D_{t(z)}(H(u, \cdot z))$ is smooth.

Nevertheless, the map $(u, z) \mapsto D_{t(z)}(H(u, \cdot z))$ needs not be compactly supported. Indeed, choose M to be a point, that is G is a genuine group. Let f be a smooth function on G, $\mu = \mathrm{d}g$ be a right invariant measure on G and $D = D_f$ be defined as in Proposition 2.3. Then for h a function on G, the map $z \mapsto \int_G f(g)h(gz)\,\mathrm{d}g$ is certainly not compactly supported in general. For example, if f is the constant function equal to 1 then the above map is the constant function equal to $\int_G h\,\mathrm{d}g$. □

Let $A \to M$ be the Lie algebroid of $G \rightrightarrows M$. Assume that we have a right invariant everywhere positive section μ of $|\Omega|^1(T^s G)$; it defines a right invariant smooth Haar system on $G \rightrightarrows M$. This section is entirely determined by its value along M in G, which is a section, denoted $\mathrm{d}\mu$, of $|\Omega|^1(A)$. Equivalently, $\mathrm{d}\mu$ is a smooth family of Lebesgue measures on the fibres of $A \to M$. By taking the dual, we obtain a family of Lebesgue measures on $A^* \to M$, the dual of $A \to M$.

Integration provides a way of embedding $C^\infty(G)$ in $C_c^\infty(G) \otimes_s C_c^\infty(M)$.

Proposition 2.3. *Let f be in $C^\infty(G)$. For each q in M, consider the following linear form on $C_c^\infty(s^{-1}(q))$*:

$$D_{f,q} : h \mapsto \int_{s^{-1}(q)} fh\mu.$$

Then D_f is in $C_c^\infty(G) \otimes_s C_c^\infty(M)$.

If moreover f has compact support, then D_f is in $\mathfrak{D}p(G)$.

Proof. Let N be a manifold and H a compactly supported smooth function on $N \times G$. It is clear that the map $(u, q) \mapsto D_{f,q}(H(u))$ has support in $(\mathrm{Id}_N \times s)(\mathrm{supp}_H)$ which is compact.

Moreover, because fH has compact support, it is a finite sum of functions with support contained in open local charts. Writing things in these local coordinates, it becomes obvious that the map $(u, q) \mapsto D_{f,q}(H(u))$ is smooth.

In addition, when f has compact support, the map:

$$N \times G^{(2)} \to C, \qquad (u, x, z) \mapsto f(x)H(u, xz)$$

has compact support. Hence

$$N \times G \to C, \qquad (u, z) \mapsto \int_{s^{-1}(t(z))} f(x)H(u, xz)$$

has compact support and D_f is in $\mathfrak{O}p(G)$. □

The Lie algebroid $\tau : A \to M$ is in particular a vector bundle and one can construct a Lie groupoid $A \rightrightarrows M$ with both the source and the target map equal to the projection $\tau : A \to M$. In particular, each smooth function on A gives an element of $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$. Let f be a smooth function on A^*. Unless the restriction of f to each fibre of A^* is L^1, the Fourier transform of f is not defined. Nevertheless, the Fourier transform of D_f is defined for a much larger class of functions.

Definition 2.4. Let f be a smooth function on A^*. Say that f has *polynomial controlled growth* if

- for every q in M,
- every smooth multi-vector field υ on M,
- every non-negative integer k and every section δ of $S^k A^*$, and
- every trivialisation $\phi : A|_{B'} \to A_q \times B'$ in a neighbourhood B' of q,

there exists a smaller neighbourhood $B \subset B'$ of q, a non-negative constant C and an integer m such that

$$(\Upsilon\Delta \cdot f)(Y) \le C(1 + \|Y\|^2)^m, \quad \text{for all } Y \text{ in } A^*|_B, \tag{1}$$

where Υ is the multi-vector field defined on A^* using υ and the trivialisation ϕ, and Δ is the multi-vector field on A^* defined using δ and the vector space structure on the fibres of A^*.

Denote by $C_{pg}{}^\infty(A^*)$ the set of smooth functions on A^* with polynomial controlled growth.

Notice that the above definition remains unchanged if one replaces $\Upsilon\Delta$ by $\Delta\Upsilon$ in (1). Also, to check if f has polynomial controlled growth, it is enough to check (1) for only one particular choice of trivialisation ϕ.

The interesting thing about functions with polynomial controlled growth is that one can define the Fourier transform of the operator D_f associated to them. This will be a consequence of the following easy lemma.

Lemma 2.5. *Let (t, ξ) be coordinates on $\mathbb{R} \times \mathbb{R}^n$ and K a compactly supported smooth function on $\mathbb{R} \times \mathbb{R}^n$. If P is any polynomial function on $\mathbb{R}^n$ then the map*:

$$\mathbb{R} \times \mathbb{R}^n \to C, \qquad (t, X) \mapsto P(X) \int_{\mathbb{R}^n} \mathrm{d}\xi \, \mathrm{e}^{-\mathrm{i}\langle X, \xi\rangle} K(t, \xi)$$

is bounded.

Proof. This is just another simple application of the fact that the Fourier transform takes multiplication by a variable to differentiation with respect to that variable. □

Corollary 2.6. *If f has polynomial controlled growth, set the Fourier transform of D_f to be*

$$\mathfrak{F}(D_f)_q(h) = \int_{A_q^*} \mathrm{d}\mu(X) f(X) \mathfrak{F}(h)(X), \quad \forall q \in M,\ h \in C_c^\infty(\tau^{-1}(q)).$$

This is a well-defined element of $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$.

Proof. If H is a compactly supported smooth function on A, then using Lemma 2.5 it is easy to prove that $D_f(H)$ is a well-defined smooth function on Q.

In addition, the support of $D_f(H)$ is included in the image of the support of H under the projection $\tau : A \to Q$ and is therefore compact. □

In Definition 2.11, I define the set of functions acceptable for quantisation as a subset of the set of functions with polynomial controlled growth. One can then apply Proposition 2.13 to see that the set of functions with polynomial controlled growth is big enough for our purpose. Moreover, it is a Poisson algebra as the next lemma shows.

Lemma 2.7. *The set of functions with polynomial controlled growth forms a Poisson sub-algebra of $C^\infty(A^*)$.*

Proof. This is a simple consequence of Lemma 3.6. □

I now wish to put a structure of algebra on $\mathfrak{D}p(G)$.

Proposition 2.8. *Let D and E be two elements of $\mathfrak{D}p(G)$. For q in M and h in $C_c^\infty(s^{-1}(q))$, set*:

$$(D \star E)_q(h) = E_q(z \mapsto D_{t(z)}(R_z^* h)).$$

The operator $D \star E$ lies in $\mathfrak{D}p(G)$.

Proof. Let N be a manifold and H a compactly supported smooth function on $N \times G$. The function F on $N \times G$ defined by $F(u, z) = D_{t(z)}(H(u, \cdot z))$ is smooth and compactly supported because D is in $\mathfrak{D}p(G)$; therefore the function $(u, z) \mapsto E_{t(z)}(F(u, \cdot z))$ is smooth and compactly supported. □

Since G is a groupoid over M, recall that its tangent groupoid is a Lie groupoid over $\mathbb{R} \times M$ with set of arrows the union of $\mathbb{R} - \{0\} \times G$ and $\{0\} \times A$. This set is given a smooth structure in a suitable way so that its Lie algebroid, called the tangent Lie algebroid, is $\mathbb{R} \times A \to \mathbb{R} \times M$ with the obvious projection and the following bracket of sections: if $(\hbar, \zeta(q, \hbar))$ and $(\hbar, \xi(q, \hbar))$ are two sections of the tangent Lie algebroid, let $\zeta_\hbar$ and $\xi_\hbar$ denote the restrictions of ζ and ξ to a fixed $\hbar$. These are sections of $A \to M$. Then

$$[(\hbar, \zeta), (\hbar, \xi)](\hbar, q) = (\hbar, \hbar[\zeta_\hbar, \xi_\hbar](q)).$$

If η is the anchor map of $A \to M$, then the anchor map of the tangent Lie algebroid is

$$\mathbb{R} \times A \to \mathbb{R} \times TM, \qquad (\hbar, \xi) \mapsto \hbar\eta(\xi).$$

For more details on tangent groupoids, see [1].

To fix the notation, let $\tilde{G}$ be the tangent groupoid of G, with respective source and target maps *tildes* and $\tilde{t}$, and $\tilde{\tau} : \tilde{A} = \mathbb{R} \times A \to \mathbb{R} \times M$ its Lie algebroid. I will present a method to construct a map from $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$ to $C_c^\infty(\tilde{G}) \otimes_{\tilde{s}} C_c^\infty(\mathbb{R} \times M)$. Let α be a diffeomorphism from an open neighbourhood W of M in A to an open neighbourhood V of M in G such that

- $\alpha(q) = q$ for q in M,
- $s \circ \alpha = \tau$, in particular α sends A_q to $s^{-1}(q)$,
- the differential at zero of the restriction of α to A_q is the identity map from A_q to A_q.

For example such an α can be obtained from the choice of an exponential map. Let

$$\tilde{W} = \{(\hbar, X) \in \mathbb{R} \times A | \, \hbar X \in W\}$$

be an open subset in $\tilde{A}$. On it, the map:

$$\tilde{\alpha}(\hbar, X) = \begin{cases} (\hbar, \alpha(\hbar X)) & \text{for } \hbar \neq 0 \\ (0, X) & \text{for } \hbar = 0 \end{cases}$$

is a diffeomorphism onto an open neighbourhood $\tilde{V}$ of $\mathbb{R} \times M$ in $\tilde{G}$. Choose a smooth function ψ on A with support in W such that $\psi|_{(1/2)W} \equiv 1$. Define $\tilde{\psi}$ in $C^\infty(\mathbb{R} \times A)$ by $\tilde{\psi}(\hbar, X) = \psi(\hbar X)$.

Proposition 2.9. *Let D be in $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$. For $(\hbar, q)$ in $\mathbb{R} \times M$ and h in $C_c^\infty(\tilde{s}^{-1}(\hbar, q))$, set*

$$\tilde{D}_{(\hbar,q)}(h) = D_q(X \mapsto \tilde{\psi}(\hbar, X) h \circ \tilde{\alpha}(\hbar, X)).$$

The operator $\tilde{D}$ lies in $C_c^\infty(\tilde{G}) \otimes_{\tilde{s}} C_c^\infty(\mathbb{R} \times M)$.

Proof. Let N be a smooth manifold and H a compactly supported smooth function on $N \times \tilde{G}$. The function $\tilde{\psi} \circ \tilde{\alpha}^{-1}$ defined on $\tilde{V}$ can be extended, by zero, to a smooth function on the whole of $\tilde{G}$. The product of this function with H is of course with compact support in $\tilde{V}$; hence its pull back by $\tilde{\alpha}$ is compactly supported. It follows that the function:

$$(N \times \mathbb{R}) \times A \to C, \qquad ((u, \hbar), X) \mapsto \psi(\hbar X) H(u, \tilde{\alpha}(\hbar, X))$$

is well defined, smooth and compactly supported. Therefore, I can apply the operator D to it and get a compactly supported smooth function on $N \times \mathbb{R} \times M$. This proves that $\tilde{D}$ is in $C_c^\infty(\tilde{G}) \otimes_{\tilde{s}} C_c^\infty(\mathbb{R} \times M)$. $\square$

Notice that in the above proof, the function ψ is used to make sense of expressions of the type $\psi(\hbar X) H(u, \tilde{\alpha}(\hbar, X))$ even when $(\hbar, X)$ is not in $\tilde{W}$, the domain of definition of $\tilde{\alpha}$.

Let us see what happens to the product of two operators constructed as in the previous proposition at $\hbar = 0$.

Lemma 2.10. *Let D_1 and D_2 be in $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$ such that $\widetilde{D_1}$ and $\widetilde{D_2}$ are in $\mathfrak{O}p(G)$. Let H be a compactly supported smooth function on $\tilde{G}$ and q be a point of M. If H_0 denotes the restriction of H to $A \subset \tilde{G}$ then*

$$(\widetilde{D_1} \star \widetilde{D_2})_{(0,q)}(H) = (D_1 \star D_2)_q(H_0).$$

In particular, if $f_{i=1,2}$ are functions on A^* such that $\mathfrak{F}(D_{f_1})$ and $\mathfrak{F}(D_{f_2})$, respectively, $\mathfrak{F}(\tilde{D}_{f_1})$ and $\widetilde{\mathfrak{F}(D_{f_2})}$, are in $\mathfrak{O}p(A)$, respectively, $\mathfrak{O}p(\tilde{G})$, then

$$(\widetilde{\mathfrak{F}(D_{f_1})} \star \widetilde{\mathfrak{F}(D_{f_2})})_{(0,q)}(H) = \mathfrak{F}(D_{f_1 f_2})_q.$$

Proof. The first claim is true because

$$(\widetilde{D_1} \star \widetilde{D_2})_{(0,q)}(H) = D_{2,q}(Y \mapsto D_{1,q}(X \mapsto H(0, X+Y))) = (D_1 \star D_2)_q(H_0).$$

The second claim is true because

$$\begin{aligned}
&(\widetilde{\mathfrak{F}(D_{f_1})} \star \widetilde{\mathfrak{F}(D_{f_2})})_{(0,q)}(H)\\
&= (\mathfrak{F}(D_{f_1}) \star \mathfrak{F}(D_{f_2}))_q(H_0)\\
&= \int_{A_q^*} \mathrm{d}\mu(X) f_2(X) \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\langle \xi, X\rangle} \int_{A_q^*} \mathrm{d}\mu(Y) f_1(Y) \int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\langle \zeta, Y\rangle} H_0(\xi+\zeta)\\
&= \int_{A_q^*} \mathrm{d}\mu(X) f_2(X) \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\langle \xi, X\rangle} \int_{A_q^*} \mathrm{d}\mu(Y) f_1(Y) \int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\langle \zeta-\xi, Y\rangle} H_0(\zeta)\\
&= \int_{A_q^*} \mathrm{d}\mu(X) f_2(X) \mathfrak{F}(\mathfrak{F}^{\,-1}(f_1 \mathfrak{F}(H_0)))(X)\\
&= \int_{A_q^*} \mathrm{d}\mu(X) f_2(X) f_1(X) \mathfrak{F}(H_0)(X) = \mathfrak{F}(D_{f_1 f_2})_q(H_0). \quad \square
\end{aligned}$$

Of course, not every element D of $C_c^\infty(A) \otimes_\tau C_c^\infty(M)$ gives an element of $\mathfrak{O}p(\tilde{G})$ and an important problem is to be able to determine when does $\tilde{D}$ lie in $\mathfrak{O}p(\tilde{G})$? More precisely, for f in $C^\infty(A^*)$, I want to know when does $\widetilde{\mathfrak{F}(D_f)}$ lie in $\mathfrak{O}p(\tilde{G})$?

Definition 2.11 gives an answer to this question.

Let f be a smooth function on A^* such that for any q in M and any compactly supported smooth function h on A_q, the product of the restriction f_q of f to A_q^* by the Fourier transform $\mathfrak{F}(h)$ is again the Fourier transform of a compactly supported smooth function denoted by $m_f(q)h$:

$$f_q \mathfrak{F}(h) = \mathfrak{F}(m_f(q)h).$$

For N a smooth manifold and $\theta : N \to M$ a smooth map, denote by Θ the induced bundle morphism $\theta^* A \to A$.

Definition 2.11. A smooth function $\tilde{H}$ on $\theta^* A$ is said to be *sufficiently compact* if:

(i) it is in $C_{vc}^\infty(\theta^* A)$, the set of vertically compactly supported smooth functions;
(ii) for any subset K of A which is compact modulo M (that is K is closed and $K \backslash M$ has compact closure),[2] the set

$$(\mathrm{supp}(\tilde{H}) + \Theta^{\,-1} K) \cap N$$

is relatively compact. This requirement says that 'small vertical perturbations of the support of $\tilde{H}$ meet N in a compact set'.

The set of sufficiently compact functions on $\theta^* A$ is denoted by $C_{sc}^\infty(\theta^* A)$.

A smooth function f on A^* is said to be *acceptable for quantisation* if:

[2] Any compact set is compact modulo M; but there might be other compacts modulo M : $M \subset A$ itself is compact modulo M even if it is not compact.

1. f has polynomial controlled growth,
2. m_f preserves $C_c^\infty(A)$,
3. m_f preserves $C_{sc}^\infty(\theta^* A)$ for all manifolds N and smooth functions $\theta : N \to M$, where the action of m_f in (1) and (2) is defined fibre-wise.

The set of smooth functions acceptable for quantisation is denoted by $\mathfrak{Q}(A^*)$.

One reason to state the rather technical above definition is the following proposition.

Proposition 2.12. *If f is acceptable for quantisation then $\widetilde{\mathfrak{F}(D_f)}$ lies in $\mathfrak{O}p(\tilde{G})$.*

Proof. Let f be acceptable for quantisation. Let N be a manifold and $\tilde{H}$ a compactly supported smooth function on $N \times \tilde{G}$. I need to prove that the function:

$$\begin{aligned} &N \times \tilde{G} \to C, \\ &(u, \hbar, z) \mapsto \widetilde{\mathfrak{F}(D_f)}_{\hbar, t(z)}((\hbar, x) \mapsto \tilde{H}(u, (\hbar, x)(\hbar, z))) \\ &\quad = \int_{A^*_{t(z)}} \mathrm{d}\mu(X) f(X) \int_{A_{t(z)}} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-i\langle \xi, X\rangle} \psi(\hbar\xi) \tilde{H}(u, \tilde{\alpha}(\hbar, \xi)(\hbar, z)), \end{aligned}$$

is compactly supported.

Define $\theta : N \times \tilde{G} \to M$ by

$$\theta(u, \hbar, z) = t(z).$$

Let F be the function:

$$\theta^* A \to C, \qquad (u, \hbar, z, \xi) \mapsto \psi(\hbar\xi)\tilde{H}(u, \tilde{\alpha}(\hbar, \xi)(\hbar, z)).$$

I need to prove that $m_f F(u, \hbar, z, 0_{t(z)})$ is compactly supported in $(u, \hbar, z)$. It will be enough to prove that F is sufficiently compact on $\theta^* A$.

Fix $(u, \hbar, z)$ in $N \times \tilde{G}$. Because $\tilde{H}$ is compactly supported and because multiplication on the right in a groupoid is a diffeomorphism between two fibres of the source map, the map:

$$N \times \tilde{s}^{\,-1}(\hbar, t(z)) \to C, \quad (u, \hbar, x) \mapsto \tilde{H}(u, (\hbar, x)(\hbar, z))$$

is compactly supported. The function $\tilde{\psi} \circ \tilde{\alpha}^{\,-1}$ is defined on an open subset of $\tilde{s}^{\,-1}(\hbar, t(z))$ and can be extended by zero to a smooth function on the whole of $\tilde{s}^{\,-1}(\hbar, t(z))$. Its product with $\tilde{H}(u, (\hbar, x)(\hbar, z))$ is compactly supported. This product composed with $\tilde{\alpha}$ is a compactly supported function on $\theta^* A_{(u,\hbar,t(z))}$. This proves that F has vertical compact support.

Let K be a compact modulo M in A. I am interested in

$$\begin{aligned} &(\mathrm{supp} F + \Theta^{\,-1} K) \cap N \times \tilde{G} \subset \{(u, \hbar, z, 0_{t(z)}) | \exists \xi \in A_{t(z)}, \\ &\hbar\xi \in \mathrm{supp}\psi,\ (u, \tilde{\alpha}(\hbar, \xi)(\hbar, z)) \in \mathrm{supp}\tilde{H},\ -\xi \in K\}. \end{aligned}$$

Let $(u_j, \hbar_j, z_j, 0)$ be a sequence in set on the right-hand side of the above inclusion. For each j, choose an element ξ_j of $-K$ such that $(u_j, \hbar_j, \tilde{\alpha}(\hbar_j, \xi_j)(\hbar_j, z_j))$ is in the support of $\tilde{H}$. This sequence satisfies

1. $-\xi_j$ is a sequence in K,
2. $\hbar_j \xi_j$ is a sequence in the support of ψ,
3. $(u_j, \hbar_j, \tilde{\alpha}(\hbar_j, \xi_j)(\hbar_j, z_j))$ is a sequence in the support of $\tilde{H}$.

Because the support of $\tilde{H}$ is compact, we can find a subsequence such that u_{j_k}, $\hbar_{j_k}$ and $\tilde{\alpha}(\hbar_{j_k}, \xi_{j_k})(\hbar_{j_k}, z_{j_k})$ converge. Also, since K is compact modulo M, we can extract a subsequence such that either ξ_{j_k} converges or ξ_{j_k} lies in M. In the former case, $\hbar_{j_k}\xi_{j_k}$ converges in suppψ and $\tilde{\alpha}(\hbar_{j_k}, \xi_{j_k})$ admits a limit, therefore $(\hbar_{j_k}, z_{j_k})$ converges. In the latter case, $\tilde{\alpha}(\hbar_{j_k}, \xi_{j_k})(\hbar_{j_k}, z_{j_k}) = (\hbar_{j_k}, z_{j_k})$ converges as well. It follows that $(\mathrm{supp}F + \Theta^{-1}K) \cap N \times \tilde{G}$ is compact. □

Proposition 2.13. *If f is either*:

- *the Fourier transform of a compactly supported smooth function on A*,
- *polynomial in the fibres, that is f is a smooth section of* $\oplus_k S^k A$,
- *a compactly supported character, i.e. it is of the type* $X_q \mapsto \mathrm{e}^{\mathrm{i}\langle \ell(q), X\rangle}$, *where* ℓ *is a compactly supported smooth section of A*,

then f is in $\mathfrak{Q}(A^*)$.

This proposition shows that $\mathfrak{Q}(A^*)$ contains indeed many interesting functions.

Proof. Let f be the Fourier transform of a compactly supported smooth function g. For each q in M, choose a local chart together with a trivialisation of A and A^* over it. Writing things in these local chart and local trivialisation, to prove that f has polynomial controlled growth is a simple matter of differentiating under the integral sign in the definition of the Fourier transform.

If f is either polynomial in the fibres or a compactly supported character, is even more immediate to prove that f has polynomial controlled growth.

Fix a compactly supported smooth function H on A, a manifold N, a smooth map $\theta : N \to M$ and a sufficiently compact smooth function $\tilde{H} \in C^\infty_{sc}(\theta^* A)$.

Firstly, assume that f is the Fourier transform of a compactly supported smooth function g on A. The support of $m_f H$ is included in the sum of supp$g$ and the support of H, hence it is compactly supported. In the same way, the support of $m_f \tilde{H}$ is included in the sum of the support of $\tilde{H}$ and Θ^{-1}supp$g$. It easily follows that $m_f \tilde{H}$ is again sufficiently compact.

Secondly, assume that f is a smooth section of $\oplus_k S^k A$. For such a function, the operator m_f is given by a differential operator ∂_f and $\partial_f H$ has support included in the support of H, therefore $\partial_f H$ is in $C^\infty_c(A^*)$. In the same way, $m_f = \partial_f$ preserves $C^\infty_{sc}(\theta^* A)$. Hence, f is acceptable for quantisation.

Finally, assume that f is of the type $f(X_q) = \mathrm{e}^{\mathrm{i}\langle \ell(q), X\rangle}$. Then the effect of m_f on H is to translate its support by ℓ on each fibre of $A \to M$. Therefore, $m_f H$ has also compact support. For the same reason, $m_f \tilde{H}$ is also still vertically compactly supported. The support of $m_f \tilde{H}$ is equal to supp$\tilde{H} + \Theta^{-1}\mathrm{Im}(-\ell)$. Since l is compactly supported, its image is compact modulo M and $m_f \tilde{H}$ is again sufficiently compact. □

Theorem 2.14. *Let* $G \rightrightarrows M$ *be a groupoid with Lie algebroid* $\tau : A \to M$. *Define a quantisation map*:

$$\mathcal{Q} : C^\infty_{pg}(A^*) \to C^\infty_c(\tilde{G}) \otimes_{\tilde{\mathfrak{F}}} C^\infty_c(\mathbb{R} \times M), \qquad f \mapsto \widetilde{\mathfrak{F}(D_f)}.$$

Then this defines a quantisation of the Poisson manifold A^* *in the sense that* $\mathcal{Q}$ *sends the set of functions acceptable for quantisation* $\mathfrak{Q}(A^*)$ *into* $\mathfrak{O}p(\tilde{G})$; *moreover, if f and g are two functions* *acceptable for quantisation then*

$$\mathcal{Q}(f) \star \mathcal{Q}(g)_{(0,q)} = \mathcal{Q}(fg)_{(0,q)}$$

and the operator $D = \frac{1}{\mathrm{i}\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$ *is in* $C_c^\infty(\tilde{G}) \otimes_{\tilde{s}} C_c^\infty(\mathbb{R} \times M)$ *with*

$$D_{(0,q)} = \mathcal{Q}(\{f, g\})_{(0,q)},$$

for every q in M.

Notice that along a non-zero $\hbar$, $\mathcal{Q}(f)$ restricts to an operator:

$$\mathcal{Q}(f)_\hbar : C_c^\infty(G) \to C_c^\infty(G),$$

while for $\hbar = 0$ it restricts to an operator $C_c^\infty(A) \to C_c^\infty(A)$ which is the Fourier transform of the operator multiplication by f on $C_c^\infty(A^*)$.

Theorem 2.14 is a consequence of Lemma 2.10 and Corollary 3.8. The proof of this corollary will take up the whole of next section.

In Section 4, I show how by applying Theorem 2.14 one recovers the quantisation of the position and momentum operators used by physicists. In Section 6, I will show how to use it to recover Weinstein strict quantisation of the torus with constant Poisson structure.

3. Computation in local coordinates

Let $m = \dim M$, U be an open subset of $\mathbb{R}^m$ and φ a diffeomorphism between U and an open subset of M:

$$\varphi : U \to \varphi(U) \subset M.$$

Let

$$U \times \mathbb{R}^n \to A|_U, \qquad (u, \xi) \mapsto \gamma(u, \xi)$$

be a trivialisation above $\varphi(U)$ of the vector bundle $A \to M$, read in the local chart (U, φ). I identify $\mathbb{R}^n$ with its dual using the usual Euclidean structure of $\mathbb{R}^n$. Therefore, γ also defines a trivialisation δ of the restriction of $A^* \to M$ to $\varphi(U)$. This trivialisation is characterised by

$$\langle \delta(u, X), \gamma(u, \xi) \rangle = \langle X, \xi \rangle,$$

where $\langle\, , \rangle$ denotes both the pairing between A and A^*, and the Euclidean product on $\mathbb{R}^n$.

Choose an open neighbourhood V of $U \times \{0\}$ in $\mathbb{R}^m \times \mathbb{R}^n$, such that α is defined on $\gamma(V)$. I can define a local chart for G:

$$\theta : V \to G, \qquad (u, v) \mapsto \alpha \circ \gamma(u, v).$$

Let V' be an open neighbourhood of $U \times \{0\}$ in V verifying:

1. for each (u, v) in V', there exists u in U such that $t \circ \theta(u, v) = \varphi(u)$,
2. for each (u_1, v_1) and (u_2, v_2) in V' with $t \circ \theta(u_2, v_2) = \varphi(u_1)$, the product $\theta(u_1, v_1)\theta(u_2, V_2)$ is in $\theta(V)$.

Let $\sigma : V' \to U$ be given by (notice that $\varphi^{-1} \circ s \circ \theta(u) = u$):

$$\sigma(u, v) = \varphi^{-1} \circ t \circ \theta(u, v).$$

Let

$$V'_{\mathrm{pr}_U} \times_\sigma V' = \{(u, v_1, v_2) \mid (u, v_2) \in V',\ (\sigma(u, v_2), v_1) \in V'\}$$

and define

$$p : V'_{\mathrm{pr}_U} \times_\sigma V' \to V, \qquad (u, v_1, v_2) \mapsto \theta^{-1}(\theta(\sigma(u, v_2), v_1)\theta(u, v_2)).$$

Ramazan [5, Proposition 2.2.5] proved:

$$p(u, v_1, v_2) = (u, v_1 + v_2 + B(u, v_1, v_2) + O_3(u, v_1, v_2)),$$
$$\theta(u, v)^{-1} = \theta(\sigma(u, v), -v + B(u, v, v) + O_3(u, v)),$$

where $B(u, v_1, v_2)$ is bilinear in (v_1, v_2) and $O_3(u, v_1, v_2)$, respectively, $O_3(u, v)$, is of degree of homogeneity at least 3 in v_1 and v_2, respectively v.

For ξ in $\mathbb{R}^n$:

$$\frac{\partial\theta}{\partial v}(u, v)\xi = \mathrm{d}_{\gamma(u,v)}\alpha \circ \frac{\partial\gamma}{\partial v}(u, v)\xi = \mathrm{d}_{\gamma(u,v)}\alpha \circ \gamma(u, \xi),$$

because γ is linear in v. In particular

$$\frac{\partial\theta}{\partial v}(u, 0)\xi = \gamma(u, \xi),$$

because $\mathrm{d}\alpha$ is the identity along M. Moreover, $\varphi(u) = \theta(u, 0)$.

The map:

$$\tilde\gamma : \mathbb{R} \times U \times \mathbb{R}^n \to \tilde A|_{\mathbb{R}\times U}, \qquad (\hbar, u, \xi) \mapsto (\hbar, \gamma(u, \xi))$$

gives a local trivialisation of the Lie algebroid $\tilde A \to \mathbb{R} \times M$ over $\mathbb{R} \times U$. Let

$$\tilde V = \{(\hbar, u, v) \mid (u, \hbar v) \in V\}$$

and

$$\widetilde{V'} = \{(\hbar, u, v) \mid (u, \hbar v) \in V'\}.$$

I obtain local coordinates on $\tilde G$ by taking $\tilde\theta = \tilde\alpha \circ \tilde\gamma$:

$$\tilde\theta : \tilde V \to \tilde G, \qquad (\hbar, u, v) \mapsto \tilde\alpha(\hbar, \gamma(u, v)).$$

Let q be in M and ξ be in A_q. Assume that ξ is in the domain of α. The map:

$$\mathcal{T}_\xi : A_q \to A_{t\circ\alpha(\xi)}, \qquad \zeta \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \alpha(\xi + r\zeta)\alpha(\xi)^{-1}$$

defines an isomorphism between A_q and $A_{t\circ\alpha(\xi)}$.

Lemma 3.1. *Let u be in U. Let $\hbar \in \mathbb{R}$ and ζ, ξ in $\mathbb{R}^n$:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} \tilde\alpha(\hbar, \mathcal{T}_{\hbar\gamma(u,\xi)} \circ \gamma(u, \zeta))\tilde\alpha(\hbar, \gamma(u, \xi)) = \mathrm{d}_{(0,u,\zeta+\xi)}\tilde\theta(1, 0, 0).$$

Proof. I first compute

$$\tilde\alpha\left(\hbar, \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \alpha \circ \gamma(u, \hbar\xi + r\zeta)\alpha \circ \gamma(u, \hbar\xi)^{-1}\right) \tilde\alpha(\hbar, \gamma(u, \xi))$$
$$= \tilde\alpha\left(\hbar, \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \theta(u, \hbar\xi + r\zeta)\theta(u, \hbar\xi)^{-1}\right) \tilde\theta(\hbar, u, \xi)$$

$$= \tilde{\alpha}\left(\hbar, \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \theta(u, \hbar\xi + r\zeta)\theta(\sigma(u, \hbar\xi), -\hbar\xi + O(\hbar^2))\right)\tilde{\theta}(\hbar, u, \xi)$$

$$= \tilde{\alpha}\left(\hbar, \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \theta(\sigma(u, \hbar\xi), r\zeta + O(\hbar^2) + B(\sigma(u, \hbar\xi), \hbar\xi + r\zeta, -\hbar\xi + O(\hbar^2)) \right.$$
$$\left. + O_3(\sigma(u, \hbar\xi), \hbar\xi + r\zeta, -\hbar\xi + O(\hbar^2)))\right)\tilde{\theta}(\hbar, u, \xi)$$

$$= \underbrace{\tilde{\theta}\circ\tilde{\gamma}^{\,-1}\left(\hbar, \frac{\partial\theta}{\partial v}(\sigma(u, \hbar\xi), 0)(\zeta + B(\sigma(u, \hbar\xi), \zeta, -\hbar\xi + O(\hbar^2)) + O(\hbar^2))\right)\tilde{\theta}(\hbar, u, \xi)}_{\text{At } r=0, \text{ we have } \theta(u,\hbar\xi+r\zeta)\theta(u,\hbar\xi)^{-1} = t\circ\theta(u,\xi), \text{ thus the differential of } \theta(u,\hbar\xi+r\zeta)\theta(u,\hbar\xi)^{-1} \text{ at } r=0 \text{ is of the type } (\partial\theta/\partial v)(\sigma(u,\hbar\xi),0)\phi \text{ for a certain vector } \phi.}$$

$$= \tilde{\theta}(\hbar, \sigma(u, \hbar\xi), \zeta - \hbar B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^2))\tilde{\theta}(\hbar, u, \xi)$$

$$= \underbrace{(\hbar, \theta(\sigma(u, \hbar\xi), \hbar\zeta - \hbar^2 B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^3))\theta(u, \hbar\xi))}_{\text{This is true only for } \hbar\neq 0, \text{ nevertheless the final result of the computation is trivially true for } \hbar=0}$$

$$= (\hbar, \theta(u, \hbar\zeta - \hbar^2 B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^3) + \hbar\xi$$
$$+ B(u, \hbar\zeta - \hbar^2 B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^3), \hbar\xi) + O(\hbar^3)))$$
$$= (\hbar, \theta(u, \hbar\zeta + \hbar\xi - \hbar^2 B(\sigma(u, \hbar\xi), \zeta, \xi) + B(u, \hbar\zeta, \hbar\xi) + O(\hbar^3)))$$
$$= \tilde{\theta}(\hbar, u, \zeta + \xi - \hbar B(\sigma(u, \hbar\xi), \zeta, \xi) + \hbar B(u, \zeta, \xi) + O(\hbar^2)).$$

The lemma follows by differentiation with respect to $\hbar$ at 0. □

Lemma 3.2. *Let f and g be in $\mathfrak{Q}(A^*)$. Let q be a point in M and H a compactly supported smooth function on $\tilde{G}$. Let*

$$N_{fg}(\hbar) = \widetilde{\mathfrak{F}(D_f)} \star \widetilde{\mathfrak{F}(D_g)}_{\hbar,q}(H),$$

then

$$\frac{\mathrm{d}N_{fg}}{\mathrm{d}\hbar}(0) = \mathfrak{F}(D_{f'g})_q(H_0) + \mathfrak{F}(D_{fg})_q\left(\left.\frac{\partial H}{\partial\hbar}\right|_{\hbar=0}\right),$$

where

$$f'(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} f(Y\circ\mathcal{T}_{\hbar\xi}^{-1})$$

and $H_0 = H|_{\hbar=0}$.

The term $\frac{\partial H}{\partial\hbar}\big|_{\hbar=0}$ is defined by first pulling back H to a neighbourhood of $\{0\}\times A$ in $\mathbb{R}\times A$ via $\tilde{\alpha}$, then differentiating with respect to $\hbar$ and finally pushing forward the result via $\tilde{\alpha}$ again. This definition is actually independent of the choice of α.

Proof. We have

$$N_{fg}(\hbar) = \int_{A_q^*}\mathrm{d}\mu(X)g(X)\int_{A_q}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle}\int_{A^*_{t\circ\alpha(\hbar\xi)}}\mathrm{d}\mu(Y)f(Y)$$
$$\times\int_{A_{t\circ\alpha(\hbar\xi)}}\mathrm{d}\mu(\zeta)\mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle}\psi(\hbar\xi)\psi(\hbar\zeta)H(\tilde{\alpha}(\hbar,\zeta)\tilde{\alpha}(\hbar,\xi)).$$

The following change of variables:

- replace Y by $Y \circ \mathcal{T}_{\hbar\xi}^{-1}$ with $Y \in A_q^*$,
- replace ζ by $\mathcal{T}_{\hbar\xi}(\zeta)$ with $\zeta \in A_q$,

gives

$$N_{fg}(\hbar) = \int_{A_q^*} \mathrm{d}\mu(X) g(X) \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle} \int_{A_q^*} \mathrm{d}\mu(Y) f(Y \circ \mathcal{T}_{\hbar\xi}^{-1})$$
$$\times \int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle} \psi(\hbar\xi)\psi(\hbar\mathcal{T}_{\hbar\xi}(\zeta)) H(\tilde{\alpha}(\hbar, \mathcal{T}_{\hbar\xi}(\zeta))\tilde{\alpha}(\hbar, \xi)).$$

The changes of variables require to introduce the terms $\det \mathcal{T}_{\hbar\xi}$ and $\det \mathcal{T}_{\hbar\xi}^{-1}$ in the above integral; but these two terms cancel each other since their product is 1.

Leaving out the justification for it for later, I differentiate the above expression under the integral signs. Since ψ is constant and equal to 1 in a neighbourhood of M, it follows that:

- $\psi(0_q) = 1$ and
- $\frac{\mathrm{d}}{\mathrm{d}\hbar}\Big|_{\hbar=0} \psi(\hbar\xi) = \frac{\mathrm{d}}{\mathrm{d}\hbar}\Big|_{\hbar=0} \psi(\hbar\mathcal{T}_{\hbar\xi}(\zeta)) = 0.$

Because of Lemma 3.1 and by definition of $\frac{\partial H}{\partial\hbar}$:

$$\frac{\mathrm{d}}{\mathrm{d}\hbar}\Big|_{\hbar=0} H(\tilde{\alpha}(\hbar, \mathcal{T}_{\hbar\xi}(\zeta))\tilde{\alpha}(\hbar, \xi)) = \frac{\partial H}{\partial\hbar}(0, \zeta + \xi).$$

The lemma follows since

$$\int_{A_q^*} \mathrm{d}\mu(X) g(X) \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle} \int_{A_q^*} \mathrm{d}\mu(Y) f(Y) \int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle} \frac{\partial H}{\partial\hbar}(0, \zeta + \xi)$$
$$= \mathfrak{F}(D_f) \star \mathfrak{F}(D_g)_q \left(\frac{\partial H}{\partial\hbar}\Big|_{\hbar=0} \right) = \mathfrak{F}(D_{fg})_q \left(\frac{\partial H}{\partial\hbar}\Big|_{\hbar=0} \right),$$

where the last line is true by Lemma 2.10.

There now remains to justify differentiation below the integral signs in

$$\int_{A_q^*} \mathrm{d}\mu(X) g(X) \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle} \int_{A_q^*} \mathrm{d}\mu(Y) f(Y \circ \mathcal{T}_{\hbar\xi}^{-1}),$$
$$\int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle} \psi(\hbar\xi)\psi(\hbar\mathcal{T}_{\hbar\xi}(\zeta)) H(\tilde{\alpha}(\hbar, \mathcal{T}_{\hbar\xi}(\zeta))\tilde{\alpha}(\hbar, \xi)).$$

Let θ be the map:

$$\tilde{V} \to M, \qquad (\hbar, \xi) \mapsto t \circ \alpha(\hbar\xi).$$

Define a function $\tilde{H}$ on $\theta^* A$ by

$$\tilde{H}(\hbar, \xi, \zeta) = \psi(\hbar\xi)\psi(\hbar\zeta) H(\tilde{\alpha}(\hbar, \zeta)\tilde{\alpha}(\hbar, \xi)).$$

Let

$$S_1(\hbar, \xi, Y) = \int_{A_q} \mathrm{d}\mu(\zeta)\, \mathrm{e}^{-\mathrm{i}\langle Y, \zeta\rangle} \tilde{H}(\hbar, \xi, \zeta),$$

$$S_2(\hbar, \xi) = \int_{A_q^*} \mathrm{d}\mu(Y) f(Y \circ \mathcal{T}_{\hbar\xi}^{-1}) S_1(\hbar, \xi, Y), \qquad S_3(\hbar, X) = \int_{A_q} \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\mathrm{i}\langle X, \xi\rangle} S_2(\hbar, \xi),$$

where S_1 is defined on $\theta^* A^*$, S_2 is defined on $\tilde{V}$ and extended by zero to $\mathbb{R} \times A$ and S_3 is defined on $\mathbb{R} \times A^*$.

I claim that $\tilde{H}$ is in $C^\infty_{sc}(\theta^* A)$. The proof of this claim is similar to that for F in the proof of Proposition 2.12 and will not be reproduced here. It follows that

$$S_2(\hbar, \xi) = m_f \tilde{H}(\hbar, \xi, 0)$$

is compactly supported in $(\hbar, \xi)$.

For ξ fixed, the function $\tilde{H}$ is compactly supported in $(\hbar, \zeta)$, hence derivation below the integral sign in S_1 is possible.

Since f has polynomial controlled growth, for ξ fixed, there exists a positive constant C, an $\epsilon > 0$ and an integer m such that

$$\left| \frac{\partial}{\partial \hbar} (f(Y \circ \mathcal{T}_{\hbar\xi}^{-1}) S_1(\hbar, \xi, Y)) \right| \leq C(1 + \|Y\|^2)^m \left(|S_1(\hbar, \xi, Y)| + \left| \frac{\partial}{\partial \hbar} S_1(\hbar, \xi, Y) \right| \right).$$

By Lemma 2.5, both terms on the right-hand side of the above inequality are bounded by a smooth L^1 function independent of $\hbar$. Differentiation below the integral sign in S_2 is therefore possible.

Since S_2 is compactly supported, differentiation below the integral sign in S_3 is possible.

To finish, since g has polynomial controlled growth and by Lemma 2.5, differentiation below the integral sign in N_{fg} is possible. □

Lemma 3.3. *Let ξ and ζ be in $\mathbb{R}^n$, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} \mathcal{T}_{\gamma(u,\hbar\xi)} \circ \gamma(u, \zeta) = \frac{\partial \gamma}{\partial u}(u, \zeta) \circ \frac{\partial \sigma}{\partial v}(u, 0)\xi - \gamma(u, B(u, \zeta, \xi)).$$

Proof. First, I compute

$$\begin{aligned}
&\mathcal{T}_{\gamma(u,\hbar\xi)} \circ \gamma(u, \zeta) \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \theta(u, \hbar\xi + r\zeta)\theta(\sigma(u, \hbar\xi), -\hbar\xi + O(\hbar^2)) \\
&\quad = \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \theta(\sigma(u, \hbar\xi), r\zeta + B(\sigma(u, \hbar\xi), \hbar\xi + r\zeta, -\hbar\xi + O(\hbar^2)) \\
&\qquad + O_3(\sigma(u, \hbar\xi), \hbar\xi + r\zeta, -\hbar\xi + O(\hbar^2))) \\
&\quad = \frac{\mathrm{d}\theta}{\mathrm{d}v}(\sigma(u, \hbar\xi), 0)(\zeta - \hbar B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^2)) \\
&\quad = \gamma(\sigma(u, \hbar\xi), \zeta - \hbar B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^2)).
\end{aligned}$$

The result follows by differentiation and because γ is linear in the second variable. □

The map $\gamma : U \times \mathbb{R}^n \to A|_U$ is a local trivialisation of A. The induced local trivialisation of A^*:

$$\delta : U \times \mathbb{R}^n \to A^*|_U$$

is characterised by

$$\langle \delta(u, X), \gamma(u, \zeta)\rangle = \langle X, \zeta\rangle,$$

where $\langle\, ,\rangle$ denotes the euclidean product on $\mathbb{R}^n$.

Lemma 3.4. *Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$, then for (u, ξ) in $U \times \mathbb{R}^n$ and Y in $A^*_{\varphi(u)}$:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} \delta^{-1}(Y \circ \mathcal{T}^{-1}_{\hbar\gamma(u\xi)}) = \left(\frac{\partial\sigma}{\partial v}(u, 0)\xi, \sum_k Y(\gamma(u, B(u, e_k, \xi)))e_k\right).$$

Proof. In the proof of Lemma 3.3, I showed that

$$\mathcal{T}_{\gamma(u,\hbar\xi)} \circ \gamma(u, \zeta) = \gamma(\sigma(u, \hbar\xi), \zeta - \hbar B(\sigma(u, \hbar\xi), \zeta, \xi) + O(\hbar^2)),$$

therefore

$$\begin{aligned}&\mathcal{T}^{-1}_{\gamma(u,\hbar\xi)} \circ \gamma(\sigma(u, \hbar\xi), \zeta)\\ &\quad = \gamma(u, \zeta + \hbar B(u, \zeta, \xi) + O(\hbar^2)) \quad \text{(Here I used}\, \sigma(u, \hbar\xi) = u + O(\hbar)\, \text{to simplify the formula).}\end{aligned}$$

The lemma is proved by using this formula when differentiating

$$\delta^{-1}(Y \circ \mathcal{T}^{-1}_{\hbar\gamma(u,\xi)}) = \left(\sigma(u, \hbar\xi), \sum_k Y(\mathcal{T}^{-1}_{\hbar\gamma(u\xi)} \circ \gamma(\sigma(u, \hbar\xi), e_k))e_k\right). \quad \square$$

Write

$$B(u, e_k, e_h) = \sum_j B^j_{k,h} e_j.$$

In particular, the B^j_{kh}'s depend on u.

Corollary 3.5. *For $Y = \sum_j Y_j e_j$ and $\xi = \sum_h \xi_h e_h$* in $\mathbb{R}^n$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} \delta^{-1}(\delta(u, Y) \circ \mathcal{T}^{-1}_{\hbar\gamma(u,\xi)}) = \left(\frac{\partial\sigma}{\partial v}(u, 0)\xi, \sum_{k,h,j} Y_j B^j_{k,h}\xi_h e_k\right).$$

Let f be a smooth function on A^*. Define $F = f \circ \delta$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} f(\delta(u, Y) \circ \mathcal{T}^{-1}_{\hbar\gamma(u,\xi)}) = \sum_{k,h} \xi_k \frac{\partial F}{\partial u_h}(u, Y)\frac{\partial\sigma_h}{\partial v_k}(u, 0) + \sum_{k,h,j} Y_j\xi_h \frac{\partial F}{\partial Y_k}(u, Y)B^j_{k,h}.$$

Proof. The first formula is just Lemma 3.4 written in local coordinates. The second one is a straightforward computation. $\square$

Let us look at the Poisson bracket on A^* in local coordinates.

Lemma 3.6. *Let f and g be smooth functions on A^*. Define $F = f \circ \delta$ and $G = g \circ \delta$, smooth functions on $U \times \mathbb{R}^n$. Set*

$$\{F, G\} = \{f, g\} \circ \delta.$$

Let (u, Z) be in $U \times \mathbb{R}^n$ and denote $\frac{\partial F}{\partial u_j}$ for $\frac{\partial F}{\partial u_j}(u, Z)$. I will use similar notations for $\frac{\partial G}{\partial u_j}$, $\frac{\partial F}{\partial Z_k}$ and $\frac{\partial G}{\partial Z_k}$. Then

$$\{F, G\}(u, Z) = \sum_{k,h,j} \frac{\partial F}{\partial Z_k}\frac{\partial G}{\partial Z_h}(B^j_{hk} - B^j_{kh})Z_j + \sum_{k,h}\left(\frac{\partial F}{\partial Z_k}\frac{\partial G}{\partial u_h} - \frac{\partial F}{\partial u_h}\frac{\partial G}{\partial Z_k}\right)\frac{\partial \sigma_h}{\partial v_k}(u, 0).$$

Proof. This is essentially Eq. (1.2.6) and Proposition 2.2.6 in Ramazan [5] where it is proved that[3]:

$$[\gamma(u, e_k), \gamma(u, e_h)] = \sum_j (B^j_{kh} - B^j_{hk})\gamma(u, e_j)$$

and, if u^*_j is the jth coordinate map on U:

$$\rho(e_k) \cdot u^*_h(u) = \frac{\partial \sigma_h}{\partial v_k}(u, 0). \qquad \square$$

Proposition 3.7. *Let f and g be in $\mathfrak{Q}(A^*)$. Let H be a compactly supported smooth function on $\tilde{G}$ and q a point in M, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0}\left(\widetilde{\mathfrak{F}(D_f)} \star \widetilde{\mathfrak{F}(D_g)} - \widetilde{\mathfrak{F}(D_g)} \star \widetilde{\mathfrak{F}(D_f)}\right)_{(\hbar,q)}(H) = \mathrm{i}\widetilde{\mathfrak{F}(D_{\{f,g\}})}_{(0,q)}(H).$$

Proof. The left-hand side of the above equation is equal to

$$\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0}(N_{fg} - N_{gf}).$$

The terms

$$\mathfrak{F}(D_{fg})_q\left(\left.\frac{\partial H}{\partial \hbar}\right|_{\hbar=0}\right)$$

appear in both $\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} N_{fg}$ and $\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} N_{gf}$ in Lemma 3.2; they will therefore cancel each other when taking the difference. The other term in $\left.\frac{\mathrm{d}}{\mathrm{d}\hbar}\right|_{\hbar=0} N_{fg}$, when using Corollary 3.5, becomes a sum of terms. These terms can be dealt with by recalling that the Fourier transform takes the operator 'multiplication by a variable' to the operator 'derivation with respect to this variable'. For example (with some slight abuse of notations):

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathrm{d}\mu(X)G(u, X)\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle}\xi_k\int_{\mathbb{R}^n}\mathrm{d}\mu(Y)\frac{\partial F}{\partial u_h}(u, Y)\frac{\partial \sigma_h}{\partial v_k}(u, 0)\\
&\quad\times\int_{\mathbb{R}^n}\mathrm{d}\mu(\zeta)\,\mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle}H_0(\xi+\zeta)\\
&= \int_{\mathbb{R}^n}\mathrm{d}\mu(X)G(u, X)\mathrm{i}\frac{\partial}{\partial X_k}\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle}\int_{\mathbb{R}^n}\mathrm{d}\mu(Y)\frac{\partial F}{\partial u_h}(u, Y)\frac{\partial \sigma_h}{\partial v_k}(u, 0)\\
&\quad\times\int_{\mathbb{R}^n}\mathrm{d}\mu(\zeta)\,\mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle}H_0(\xi+\zeta)
\end{aligned}$$

[3] The signs here and in Ramazan [5] do not agree. This is due to different choices in the definition of the map α. Essentially, I have $s \circ \alpha$ constant on the fibres of $A \to Q$, whereas he has $t \circ \alpha$ constant on the same fibres.

$$
\begin{aligned}
&= -\mathrm{i}\int_{\mathbb{R}^n} \mathrm{d}\mu(X)\frac{\partial G}{\partial X_k}(u,X)\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle}\int_{\mathbb{R}^n}\mathrm{d}\mu(Y)\frac{\partial F}{\partial u_h}(u,Y)\frac{\partial\sigma_h}{\partial v_k}(u,0)\\
&\quad\times\int_{\mathbb{R}^n}\mathrm{d}\mu(\zeta)\,\mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle}H_0(\xi+\zeta)\\
&= -\mathrm{i}\int_{\mathbb{R}^n}\mathrm{d}\mu(Z)\frac{\partial G}{\partial Z_k}(u,Z)\frac{\partial F}{\partial u_h}(u,Z)\frac{\partial\sigma_h}{\partial v_k}(u,0)\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle Z,\xi\rangle}H_0(\xi),
\end{aligned}
$$

where the last line is true for the same reason that $\mathfrak{F}(D_f)\star\mathfrak{F}(D_g)=\mathfrak{F}(D_{fg})$ (see Lemma 2.10). A similar computations leads to

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\mathrm{d}\mu(X)G(u,X)\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle X,\xi\rangle}\xi_h\int_{\mathbb{R}^n} v\mu(Y)Y_jB^j_{kh}\frac{\partial F}{\partial Y_k}(u,Y)\\
&\quad\times\int_{\mathbb{R}^n}\mathrm{d}\mu(\zeta)\,\mathrm{e}^{-\mathrm{i}\langle Y,\zeta\rangle}H_0(\xi+\zeta)\\
&= -\mathrm{i}\int_{\mathbb{R}^n}\mathrm{d}\mu(Z)\frac{\partial G}{\partial Z_h}(u,Z)\frac{\partial F}{\partial Z_k}(u,Z)Z_jB^j_{kh}\int_{\mathbb{R}^n}\mathrm{d}\mu(\xi)\,\mathrm{e}^{-\mathrm{i}\langle Z,\xi\rangle}H_0(\xi).
\end{aligned}
$$

These computations together with Lemma 3.6 prove Proposition 3.7. □

I obtain the following corollary.

Corollary 3.8. *Let f and g be two smooth functions on A^* acceptable for quantisation. Then*

$$\frac{1}{\mathrm{i}\hbar}[\widetilde{\mathfrak{F}(D_f)},\widetilde{\mathfrak{F}(D_g)}]$$

is a well-defined element of $C_c^\infty(\tilde{G})\otimes_{\mathfrak{F}}C_c^\infty(\mathbb{R}\times M)$, which along $\hbar=0$ is equal to

$$\widetilde{\mathfrak{F}(D_{\{f,g\}})}.$$

4. Quantisation of $\mathbb{R}^{2n}$, with and without a magnetic field

In this short section, I will discuss the case of the quantisation of observables on the phase space of a particle in $\mathbb{R}^n$.

Let $M=\mathbb{R}^n$ with its euclidean structure and A be the tangent bundle of M, that is $A=\mathbb{R}^{2n}$. In these conditions, the Lie groupoid G integrating A is the pair groupoid $M\times M=\mathbb{R}^n\times\mathbb{R}^n$ with source map, tangent map and product:

$$s(p,q)=q,\ t(p,q)=p,\ (r,p)\cdot(p,q)=(r,q).$$

The euclidean product gives a natural family of measures on the fibres of A. I can take α to be defined on the whole of A by

$$\alpha(q,\xi)=(q+\xi,q).$$

Also, I can choose ψ to be equal to the constant function 1. The space of morphisms of the tangent groupoid is diffeormorhpic to $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Let H be a compactly supported smooth function on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$; it is a function of $(\hbar,p,q)$. Let f be a function acceptable for quantisation.

For $\hbar \neq 0$ and $x = (p, q)$ in G:

$$\begin{aligned}\mathcal{Q}_\hbar(f)(H)(x) &= \int_{A_p^*} \mathrm{d}X\, f(p, X) \int_{A_p} \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}\langle \xi, X\rangle} H(\tilde{\alpha}(\hbar, p, \xi) \cdot (\hbar, p, q)) \\ &= \int_{\mathbb{R}^n} \mathrm{d}X\, f(p, X) \int_{\mathbb{R}^n} \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}\langle \xi, X\rangle} H((\hbar, p + \hbar\xi, p) \cdot (\hbar, p, q)) \\ &= \int_{\mathbb{R}^n} \mathrm{d}X\, f(p, X) \int_{\mathbb{R}^n} \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}\langle \xi, X\rangle} H(\hbar, p + \hbar\xi, q).\end{aligned}$$

If $f(p, X)$ is equal to X_k, the k-th coordinate of X, its quantisation at a given value of $\hbar$ is

$$\mathcal{Q}_\hbar(X_k) = -\mathrm{i}\hbar \frac{\partial}{\partial p_k},$$

whereas if $f(p, X) = p_k$ then

$$\mathcal{Q}_\hbar(p_k) = p_k,$$

the operator multiplication by p_k.

A magnetic field (see [4], p. 178) is given by a closed 2-form B on M. Since $M = \mathbb{R}^n$ has no cohomology in degree 2, it follows that there exists a one-form (the vector potential) A such that $B = \mathrm{d}A$. If the particle we are studying carries an electric charge e and has mass m, then the Hamiltonian of the system can be taken to be

$$H_A(p, X) = \frac{1}{2m} ||X - eA||^2,$$

where A is seen as a section of T^*M. This Hamiltonian is acceptable for quantisation and one can therefore quantise the system consisting of the particle moving in the magnetic field. For a different, and more detailed, approach to this problem, see Mantoiu and Purice [3].

5. Some general results about Poisson manifolds

Given a Poisson manifold P, its cotangent bundle is naturally a Lie algebroid A. If this Lie algebroid is integrable to a Lie groupoid then Theorem 2.14 gives a quantisation of the Lie–Poisson manifold A^*. Since the Poisson structure of A^* is completely determined by the one of P, on might hope to be able to say something about a quantisation of P. One way of doing so might consist in looking for a surjective Poisson map π between A^* and P and then quantised a function on P by taking the quantisation of the pulled back function on A^*. In C^*-algebraic quantisation, such an idea is bound to fail because if f is a function on P, then its pull-back $\pi^* f$ has little chance of being quantisable. Nevertheless, I will show with an example that this idea can be made to work when using the quantisation defined in Theorem 2.14.

The aim of this section is to derive the partial differential equation that a map $\pi : TP \to P$ has to satisfy to be Poisson. This PDE is given in Corollary 5.2.

Assume P is a Poisson manifold with Poisson bivector field η. Denote by $p : TP \to P$ the natural projection. The Poisson structure of P induces a Lie algebroid structure on the cotangent space of P with anchor map $\eta : T^*P \to TP$[4]. Its dual, the tangent space of P, inherits a Poisson

[4] Here, η is understood as an anti-symmetric map between T^*P and TP. I will use different sorts of interpretations of η, the precise interpretation depending on the context.

structure in the following manner. Let α and β be closed one-forms on P. They naturally define smooth functions, denoted $\tilde{\alpha}$ and $\tilde{\beta}$, on TP by, for v in T_xP:

$$\tilde{\alpha}(v) = \alpha(v), \quad \text{and } \tilde{\beta}(v) = \beta(v).$$

Put

$$\{\tilde{\alpha}, \tilde{\beta}\}_{TP}(v) = [\alpha, \beta](v), \tag{2}$$

where [,] is the bracket on the Lie algebroid T^*P. Let f and g be smooth functions on P. Put

$$\{p^* f, p^* g\}_{TP} = 0. \tag{3}$$

Finally, for v in T_xP, put

$$\{\tilde{\alpha}, p^* f\}_{TP}(v) = \eta(\alpha(x)) \cdot f. \tag{4}$$

Formulae (2)–(4) completely determine the Poisson structure on TP.

Pick a torsion free connection on TP:

$$\nabla : \Gamma(TP) \otimes \Gamma(TP) \to \Gamma(TP), \qquad (X, Y) \mapsto \nabla_X Y.$$

For example, the Levi–Civita connection of a metric would do. In particular, for every v in T_xP, there is a splitting of $T_v(TP)$ as a direct sum of a horizontal space $\mathbf{H}_v(TP)$ and a vertical space $\mathbf{V}_v(TP)$. Denote by $\mathbf{H}$ and $\mathbf{V}$ the projection on respectively $\mathbf{H}(TP)$ and $\mathbf{V}(TP)$. Both these spaces are isomorphic to T_xP and the projection $\mathbf{H}$ is equal to p_* while $\mathbf{V}(TP)$ is the kernel of p_*. The isomorphism between $\mathbf{V}_v(TP)$ and T_xP is given by

$$T_xP \to \mathbf{V}_v(TP), \qquad u \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v + tu.$$

Let μ be in $\bigwedge^2 T_xP$. The connection also defines a splitting of the tangent space of $\bigwedge^2 TP$ at μ into the direct sum of a horizontal space isomorphic to T_xP and a vertical space isomorphic to $\bigwedge^2 T_xP$. Because $p \circ \eta$ is the identity of P, the horizontal component of $T\eta : TP \to T\bigwedge^2 TP$ is the identity. Denote its vertical component by

$$\mathfrak{D}\eta : TP \to \bigwedge^2 TP.$$

If f is a smooth function on TP, its differential at v in T_xP has a horizontal and a vertical component. Denote by $\partial_2 f : T_xP \to C$ its horizontal component and by $\partial_1 f : T_xP \to C$ its vertical one. In the same fashion, for $\pi : TP \to P$, denote by $\partial_1\pi$ and $\partial_2 p$ respectively the vertical and horizontal components of $T\pi : T(TP) \to TP$.

Lemma 5.1. *Let f and g be smooth functions on TP. For x in P and v in T_xP:*

$$\{f, g\}_{TP}(v) = \langle \mathfrak{D}\eta(v), \partial_1 f(v) \wedge \partial_1 g(v)\rangle + \langle \eta(x), \partial_1 f(v) \wedge \partial_2 g(v) - \partial_1 g(v) \wedge \partial_2 f(v)\rangle.$$

Proof. Use the right-hand side of the above equation to define a bracket

$$\{,\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P).$$

This bracket satisfies the Leibniz identity because the operators ∂_1 and ∂_2 do. It is also clearly anti-symmetric. To prove that it is equal to $\{,\}_{TP}$, it suffices to prove that it satisfies Eqs. (2)–(4).

Eq. (3) is satisfied because ∂_1 vanishes on pull-backs to TP of functions on P.

If g is a function on P, then $\partial_2 p^* g = \mathrm{d}g$. If f is equal to $\tilde{\alpha}$ for some one-form α on P, then $\partial_1\tilde{\alpha}(v) = \alpha_x$ for all v in T_xP. Hence, the bracket { , } satisfies Eq. (4).

The connection on $TP \to P$ also defines a connection on its dual bundle and on all bundles one can construct from TP and T^*P through direct sums, tensor products, etc. Since the definition of a Poisson structure is local, I can assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are exact forms when checking that $\{\}$ satisfies Eq. 2. Let α and β be closed one-forms on P. Let v be in T_xP and let σ be a path in P such that $\sigma(0) = x$ and $\dot{\sigma}(0) = v$; for example, take $\sigma(t) = \mathrm{Exp}(tv)$, where Exp is the exponential map of the connection. Eq. (2) gives

$$\{\tilde{\alpha}, \tilde{\beta}\}_{TP}(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta_{\sigma(t)}(\alpha \wedge \beta). \tag{5}$$

Let $A \to P$ be the bundle $T^*P \oplus T^*P \oplus \bigwedge^2 TP \to P$. There is a natural map:

$$m : A \to \mathbb{R} \qquad (\alpha, \beta, \mu) \mapsto \mu(\alpha \wedge \beta)$$

I will compute the differential of this map. Let w be in $T_{(\alpha_x, \beta_x, \mu_x)}A$ with horizontal component $\mathbf{H}(w) = v$. Its vertical component $\mathbf{V}(w) = (\epsilon_1, \epsilon_2, \theta)$ is in $T_x^*P \oplus T_x^*P \oplus \wedge^2 T_xP$. Denote $\phi_{\sigma(t)} : A_x \to A_{\sigma(t)}$ be the parallel transport along the path σ. Define a path in A by

$$\gamma(t) = \phi_{\sigma(t)}(\alpha_x + t\epsilon_1, \beta_x + t\epsilon_2, \mu_x + t\theta).$$

The path γ satisfies

$$\gamma(0) = (\alpha_x, \beta_x, \mu_x), \quad \text{and } \dot{\gamma}(0) = w.$$

Notice that, because the connection on A is defined using a single connection on TP, we have

$$m \circ \phi_{\sigma(t)} = m.$$

This means

$$\begin{aligned} m_*(w) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} m \circ \phi_{\sigma(t)}(\alpha_x + t\epsilon_1, \beta_x + t\epsilon_2, \mu_x + t\theta) \\ &= m(\epsilon_1, \beta_x, \mu_x) + m(\alpha_x, \epsilon_2, \mu_x) + m(\alpha_x, \beta_x, \theta). \end{aligned}$$

This last computation together with Eq. (5) gives

$$\{\tilde{\alpha}, \tilde{\beta}\}_{TP}(v) = m(\partial_{1,x}\alpha(v), \beta_x, \eta_x) + m(\alpha_x, \partial_{1,x}\beta(v), \eta_x) + m(\alpha_x, \beta_x, \mathfrak{D}\eta(v)). \tag{6}$$

Firstly, in this equality, one can replace α_x and β_x by respectively $\partial_{1,v}\tilde{\alpha}$ and $\partial_{1,v}\tilde{\beta}$. Secondly, consider a vector field X on P. Let

$$\iota : P \to T^*P \oplus TP, \qquad x \mapsto (\alpha_x, X_x),$$

and

$$k : T^*P \oplus TP \to C, \qquad (\delta, Z) \mapsto \delta(Z).$$

I choose X such that $\partial_1 X = 0$. Then, differentiation of the equality:

$$k \circ \iota = \tilde{\alpha} \circ X$$

leads to

$$\partial_{2,X_x}\tilde{\alpha}(v) = \partial_{1,x}\alpha(v)(X_x). \tag{7}$$

Since α and β are closed one-forms, and since ∇ is torsion free:

$$\partial_{1,x}\alpha(v)(X_x) = \partial_{1,x}\alpha(X_x)(v). \tag{8}$$

Eqs. (6)–(8) put together prove that the bracket $\{,\}$ satisfies Eq. (2). $\square$

I deduce the following corollaries.

Corollary 5.2. *Let π be a map $TP \to P$. Let f and g be functions on P. Their pull-backs by π satisfy*

$$\{\pi^* f, \pi^* g\}_{TP}(v) = \langle (\partial_1\pi(v) \otimes \partial_1\pi(v))(\mathfrak{D}_x\eta(v)) + 2(\partial_1\pi(v) \otimes \partial_2\pi(v))(\eta(x)), \mathrm{d}_{\pi(v)} f \wedge d_{\pi(v)} g \rangle.$$

In particular, π is Poisson if and only if

$$\frac{1}{2}(\partial_1\pi(v) \odot \partial_1\pi(v))(\mathfrak{D}_x\eta(v)) + (\partial_1\pi(v) \odot \partial_2\pi(v))(\eta(x)) = \eta(\pi(v)),$$

for all v in T_xP, where $\odot$ means the symmetric product.

Corollary 5.3. *Assume the Poisson bivector field is parallel relative to the connection ∇. Then, a map $\pi : TP \to P$ is Poisson if and only if*

$$(\partial_1\pi(v) \odot \partial_2\pi(v))(\eta(x)) = \eta(\pi(v)),$$

for all v in T_xP.

Proof. Indeed, since the Poisson bivector field is parallel:

$$\mathfrak{D}\eta(v) = 0. \qquad \square$$

Assume the connection on P is the Levi–Civita connection of a metric on P and assume P is complete. Its exponential map is denoted

$$\mathrm{Exp} : TP \to P.$$

When restricted to a fibre T_xP, I will denote it Exp_x. I quote here the following lemma for future reference.

Lemma 5.4. *Let v be a tangent vector to P at a point x. Let w be a tangent vector to TP at v. Its horizontal and vertical components are respectively* $\mathbf{H}w$ and $\mathbf{V}w$. *Consider the geodesic* $\sigma(t) = \mathrm{Exp}(t\mathbf{H}w)$ *and the one-parameter family of geodesics*:

$$\gamma_s(t) = \mathrm{Exp}_{\sigma(t)}(s\phi_{\sigma(t)}(v + t\mathbf{V}w)),$$

where ϕ is the parallel transport. The differential of Exp *at v is given by*

$$\mathrm{T}_v\mathrm{Exp}(w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_1(t).$$

It follows that $\mathrm{T}_v\mathrm{Exp}(w)$ *is also the value at $t = 1$ of the Jacobi field J along the geodesic σ with initial value $J(0) = \mathbf{H}w$ and $J'(0) = \mathbf{V}w$. In particular,* $\mathrm{Exp} : \mathrm{T}P \to P$ *is a submersion.*

Proof. This is a simple exercise in Riemannian geometry. $\square$

6. The torus with constant Poisson structure

Let P be the n-dimensionnal torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ with its metric inherited from the euclidean metric $\langle , \rangle$ on $\mathbb{R}^n$. Consider a constant Poisson structure on P given by a skew-symmetric $n \times n$ matrix

η. Identify $\mathrm{T}P$ with $\mathbb{R}^n \times P$ in the obvious way. From Corollary 5.2, I deduce that the map:

$$\pi : \mathbb{R}^n \times P \to P, \qquad (u, p) \mapsto \mathrm{Exp}_p\left(\frac{1}{2}u\right) = p + \frac{1}{2}u$$

is Poisson. It is also a surjective submersion. Hence, I can hope that a quantisation of $\mathrm{T}P$ will lead to a quantisation of P.

The dual $A = T^*P$ of $\mathrm{T}P$, identified with $\mathrm{T}P$ using the euclidean metric, is a Lie algebroid. It can be integrated to a source simply connected Lie groupoid. The space of morphims of this groupoid is $\mathbb{R}^n \times P$. Notice that $\mathbb{R}^n$ is the direct orthogonal sum of $\mathrm{Ker}(\eta)$ and $\mathrm{Im}(\eta)$. Let pr_1 be the orthogonal projection on $\mathrm{Ker}(\eta)$ and pr_2 the orthogonal projection on $\mathrm{Im}(\eta)$. The source map of the groupoid is

$$s : \mathbb{R}^n \times P \to P, \qquad (u, p) \mapsto p,$$

whereas the target map is

$$t : \mathbb{R}^n \times P \to P, \qquad (u, p) \mapsto p + \mathrm{pr}_2(u).$$

Given two elements (u, p) and (v, q) in the groupoid, their multiplication $(u, p) \cdot (v, q)$ is well defined if the target of (u, p) is equal to the source of (v, q), that is if $q = p + \mathrm{pr}_2(u)$; in this case

$$(u, p) \cdot (v, q) = (u + v, p).$$

Assume that η is invertible, that is P is symplectic. In this case, η defines an isomorphism of Lie algebroids between T^*P and TP:

$$T^*P \to TP, \qquad (\xi, p) \mapsto (\eta(\xi), p).$$

Choose the natural connection on the trivial vector bundle $TP \simeq \mathbb{R}^n \times P \to P$.

The exponential map Exp for TP is

$$TP \to \mathbb{R}^n \times P, \qquad (X, p) \mapsto (X, p).$$

Whereas the Exp map for T^*P, that is α, is

$$T^*P \to P, \qquad (\xi, p) \mapsto (\eta(\xi), p).$$

The tangent groupoid is given by $\tilde{G} = \mathbb{R} \times \mathbb{R}^n \times P$ with

$$s(\hbar, u, q) = (\hbar, q), \qquad \text{and } t(\hbar, u, q) = (\hbar, q + \hbar u).$$

The product is given by

$$(\hbar, v, q + \hbar u) \cdot (\hbar, u, q) = (\hbar, v + u, q).$$

With this representation of $\tilde{G}$, the exponential of the tangent groupoid is

$$\mathbb{R} \times TP \to \mathbb{R} \times \mathbb{R}^n \times P, \qquad (\hbar, X, q) \mapsto (\hbar, X, q),$$

therefore $\tilde{\alpha}$ is

$$\mathbb{R} \times T^*P \to \mathbb{R} \times \mathbb{R}^n \times P, \qquad (\hbar, \xi, q) \mapsto (\hbar, \eta(\xi), q).$$

Let f be a function on $A^* = TP$. Assume it is acceptable for quantisation. Let H be a compactly supported smooth function on $\tilde{G}$. For $(\hbar, q)$ in $\mathbb{R} \times P$:

$$\mathcal{Q}(f)_{\hbar,q}(H) = \int_{\mathbb{R}^n} \mathrm{d}(X)\, f(X, q) \int_{\mathbb{R}^n} \mathrm{d}(\xi)\, \mathrm{e}^{-\mathrm{i}\langle \xi, X\rangle} H(\hbar, \eta(\xi), q). \tag{9}$$

Let r be a vector in $\mathbb{Z}^n$ and define a function:

$$g_r : P \to C, \qquad q \mapsto \mathrm{e}^{\mathrm{i}\langle r,q\rangle}.$$

The number $\mathrm{e}^{\mathrm{i}\langle r,q\rangle}$ is well defined because $\langle r, q\rangle$ is well-defined modulo 2π. Set $f_r = \pi^* g_r$, that is

$$f_r : TP \to C, \qquad (X, q) \mapsto \mathrm{e}^{\mathrm{i}\langle r,q\rangle}\mathrm{e}^{1/2\mathrm{i}\langle r,X\rangle}.$$

Proposition 6.1. *Let r and r′ be vectors in* $\mathbb{Z}^n$, *then*

$$\mathcal{Q}(f_r) \star \mathcal{Q}(f_{r'}) = \mathcal{Q}(\mathrm{e}^{\mathrm{i}\hbar/2\langle r,\eta(r')\rangle} f_{r+r'}).$$

Proof. With f_r instead of f, (9) becomes

$$\begin{aligned}\mathcal{Q}(f_r)_{\hbar,q}(H) &= \int_{\mathbb{R}^n} \mathrm{d}(X)\, \mathrm{e}^{\mathrm{i}\langle r,q\rangle}\, \mathrm{e}^{1/2\mathrm{i}\langle r,X\rangle} \int_{\mathbb{R}^n} \mathrm{d}(\xi)\, \mathrm{e}^{-\mathrm{i}\langle \xi,X\rangle} H(\hbar, \eta(\xi), q)\\ &= \mathrm{e}^{\mathrm{i}\langle r,q\rangle} H(\hbar, \eta(1/2r), q).\end{aligned}$$

Hence, the product $\mathcal{Q}(f_r) \star \mathcal{Q}(f_{r'})$ is

$$\begin{aligned}&\mathcal{Q}(f_r) \star \mathcal{Q}(f_{r'})_{\hbar,q}(H)\\ &\quad= \int_{\mathbb{R}^n} \mathrm{d}(X)\ f_{r'}(X, q) \int_{\mathbb{R}^n} \mathrm{d}(\xi)\, \mathrm{e}^{\mathrm{i}\langle \xi,X\rangle} \int_{\mathbb{R}^n} \mathrm{d}(Y) f_r(Y, q + \hbar\eta(\xi))\\ &\qquad\times \int_{\mathbb{R}^n} \mathrm{d}(\zeta)\, \mathrm{e}^{-\mathrm{i}\langle \zeta,Y\rangle} H(\hbar, \eta(\zeta + \xi), q)\\ &\quad= \int_{\mathbb{R}^n} \mathrm{d}(X)\, \mathrm{ei}\langle r', q\rangle\, \mathrm{e}^{1/2\mathrm{i}\langle r',X\rangle} \int_{\mathbb{R}^n} \mathrm{d}(\xi)\mathrm{e}^{\mathrm{i}\langle \xi,X\rangle} \int_{\mathbb{R}^n} \mathrm{d}(Y)\mathrm{e}^{\mathrm{i}\langle r,q+\hbar\eta(\xi)\rangle}\, \mathrm{e}1/2\mathrm{i}\langle r, Y\rangle\\ &\qquad\times \int_{\mathbb{R}^n} \mathrm{d}(\zeta)\mathrm{e}^{-\mathrm{i}\langle \zeta,Y\rangle} H(\hbar, \eta(\zeta + \xi), q)\\ &\quad= \mathrm{e}^{\mathrm{i}\langle r'+r,q\rangle} \int_{\mathbb{R}^n} \mathrm{d}(Y)\mathrm{e}^{\mathrm{i}\langle r,\hbar\eta(1/2r')\rangle}\, \mathrm{e}^{\mathrm{i}\langle r,Y\rangle} \int_{\mathbb{R}^n} \mathrm{d}(\zeta)\mathrm{e}^{-\mathrm{i}\langle \zeta,Y\rangle} H(\hbar, \eta(\zeta + 1/2r'), q)\\ &\quad= \mathrm{e}^{\mathrm{i}\langle r'+r,q\rangle}\, \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}\langle r,\eta(r')\rangle} H(\hbar, 1/2\eta(r + r'), q) = \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}\langle r,\eta(r')\rangle} \mathcal{Q}(f_{r+r'})_{\hbar,q}(H). \qquad \square\end{aligned}$$

Let $\mathcal{P}$ be the algebra of functions on P generated by $\{g_r,\ r \in \mathbb{Z}^n\}$. It is a dense subalgebra of the C^*-algebra of continuous functions on P. Proposition 6.1 shows that the product on this algebra can be deformed in

$$g_r \star_\hbar g_{r'} = \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}\langle r,\eta(r')\rangle} g_{r+r'},$$

for each $\hbar$. With this new product, $\mathcal{P}$ becomes a *-algebra which can be completed into a C^*-algebra $\mathcal{P}_\hbar$. The natural family of injections $\mathcal{P} \to \mathcal{P}_\hbar$ gives the usual quantisation of the torus with constant Poisson structure as defined in Tang and Weinstein [7]. It is a strict deformation quantisation in the sense of Rieffel.

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Appendix A. The two-sphere in $\mathbb{R}^3$

Let $P = S^2$ be the two-sphere $\{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1\}$. In this section, I show how to construct a Poisson map between TS^2 and S^2.

Consider the metric on S^2 given by the restriction of the euclidean metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$. For $p = (x, y, z)$ in $\mathbb{R}^3$, define the endomorphism:

$$J_p(u) = u \wedge p, \quad \text{for all } u \in \mathbb{R}^3.$$

The restriction of J to each tangent space of S^2 defines a complex structure on the sphere. Also

$$\omega = g(J\cdot, \cdot)$$

is a symplectic form on S^2. It is the restriction to S^2 of the two-form $z\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y$ defined on $\mathbb{R}^3$. The geodesics are great circles on the sphere so that the exponential is given by

$$\mathrm{Exp}_p(u) = p\cos(\|u\|) + \frac{\sin(\|u\|)}{\|u\|}u.$$

The differential $\partial_1\mathrm{Exp}$ is given by

$$\begin{aligned}\partial_1\mathrm{Exp}_{(u,p)}(h) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_p(u + th)\\ &= \frac{\sin(\|u\|)}{\|u\|}(-\langle u, h\rangle p + h) + \frac{\|u\|\cos(\|u\|) - \sin(\|u\|)}{\|u^2}\| \left\langle \frac{u}{\|u\|}, h\right\rangle u,\end{aligned}$$

this is a map from T_pS^2 to $\mathrm{T}_{\mathrm{Exp}_p(u)}S^2$.

Also

$$\partial_2\mathrm{Exp}_{(u,p)}(\epsilon) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_{\sigma(t)}\phi_{\sigma(t)}(u),$$

where $\sigma(t) = \mathrm{Exp}_p(t\epsilon)$ and $\phi_{\sigma(t)}$ is the parallel transport along $\sigma(t)$. Without loss of generality, I can assume that ϵ is a unit vector. In this situation:

$$\sigma(t) = \cos(t)p + \sin(t)\epsilon.$$

Also, $\{p, \epsilon, p \wedge \epsilon\}$ forms an orthonormal basis of $\mathbb{R}$ and $\phi_{\sigma(t)}$ is a morphism in SO(3). It is given by

$$\phi_{\sigma(t)}(p) = \sigma(t), \qquad \phi_{\sigma(t)}(\epsilon) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t), \qquad = -\sin(t)p + \cos(t)\epsilon, \qquad \phi_{\sigma(t)}(p \wedge \epsilon) = p \wedge \epsilon.$$

It follows that

$$\phi_{\sigma(t)}(u) = \langle u, \epsilon\rangle(-\sin(t)p + \cos(t)\epsilon) + \langle u, p \wedge \epsilon\rangle p \wedge \epsilon.$$

The parallel transport preserves the norm, hence u and $\phi_{\sigma(t)}(u)$ have the same norm and

$$\begin{aligned}\mathrm{Exp}_{\sigma(t)}\phi_{\sigma(t)}(u) &= \cos(\|u\|)\sigma(t) + \frac{\sin(\|u\|)}{\|u\|}\phi_{\sigma(t)}(u)\\ &= \cos(\|u\|)(\cos(t)p + \sin(t)\epsilon) + \frac{\sin(\|u\|)}{\|u\|}(\langle u, \epsilon\rangle(-\sin(t)p\\ &\quad + \cos(t)\epsilon) + \langle u, p \wedge \epsilon\rangle p \wedge \epsilon).\end{aligned}$$

I can now compute

$$\partial_2 \mathrm{Exp}_{(u,p)}(\epsilon) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_{\sigma(t)}\phi_{\sigma(t)}(u) = \cos(\|u\|)\epsilon - \frac{\sin(\|u\|)}{\|u\|}\langle u, \epsilon\rangle p.$$

This last formula is of course still valid when ϵ is not a unit vector.

Consider a map

$$\pi : \mathrm{T}S^2 \to S^2 \qquad u \mapsto \mathrm{Exp}(\lambda \|u\|) = \cos(\lambda\|u\|)p + f(\lambda\|u\|)\lambda u,$$

where λ is a function of $\|u\|^2$ defined for $\|u\|^2$ in some neighbourhood of 0 in $\mathbb{R}$. For π to be a Poisson map, I need $\lambda(0) = \frac{1}{2}$.

I will now compute the differential of such a map. Firstly

$$\begin{aligned}
\partial_{1,(u,p)}\pi(h) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_p(\lambda(\|u+th\|^2)(u+th))\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \cos(\lambda(\|u+th\|^2)\,\|u+th\|)p\\
&\quad + f(\lambda(\|u+th\|^2)\,\|u+th\|)\lambda(\|u+th\|^2)(u+th)\\
&= -\sin(\lambda\,\|u\|)(2\lambda'\langle u,h\rangle\,\|u\| + \lambda\langle h, \frac{u}{\|u\|}\rangle)p + f(\lambda\,\|u\|)\lambda h\\
&\quad + 2f(\lambda\,\|u\|)\langle u,h\rangle\lambda' u + (2\langle u,h\rangle\lambda'\,\|u\| + \lambda\langle h, \frac{u}{\|u\|}\rangle)f'(\lambda\,\|u\|)\lambda u\\
&= -\lambda f(\lambda\,\|u\|)(2\lambda'\langle u,h\rangle\,\|u\|^2 + \lambda\langle u,h\rangle)\lambda f(\lambda\,\|u\|)p + f(\lambda\,\|u\|)\lambda h\\
&\quad + (2f(\lambda\,\|u\|)\lambda' + 2\lambda'\lambda f'(\lambda\,\|u\|) + \frac{\lambda^2}{\|u\|}f'(\lambda\,\|u\|))\langle u,h\rangle u\\
&= -\lambda f(\lambda\,\|u\|)(2\lambda'\langle u,h\rangle\,\|u\|^2 + \lambda\langle u,h\rangle)\lambda f(\lambda\,\|u\|)p + f(\lambda\,\|u\|)\lambda h\\
&\quad + (2\lambda'\cos(\lambda\,\|u\|) + \frac{\lambda^2}{\|u\|}f'(\lambda\,\|u\|))\langle u,h\rangle u,
\end{aligned}$$

where for the last equality I have used the relation:

$$tf'(t) = \cos(t) - f(t).$$

Secondly, assuming without loss of generality that $\|\epsilon\| = 1$:

$$\begin{aligned}
\partial_{2,(u,p)}\pi(\epsilon) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\phi_{\sigma(t)}(u))\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_{\sigma(t)}(\lambda(\left\|\phi_{\sigma(t)}(u)\right\|^2)\phi_{\sigma(t)}(u))\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Exp}_{\sigma(t)}(\lambda(\|(u)\|^2)\phi_{\sigma(t)}(u))\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \cos(\lambda\left\|\phi_{\sigma(t)}(u)\right\|)\sigma(t) + f(\lambda\left\|\phi_{\sigma(t)}(u)\right\|)\lambda\phi_{\sigma(t)}(u)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \cos(\lambda\,\|u\|)\sigma(t) + f(\lambda\,\|u\|)\lambda\phi_{\sigma(t)}(\langle u,\epsilon\rangle\epsilon + \langle u, p\times\epsilon\rangle p\times\epsilon)
\end{aligned}$$

$$= \cos(\lambda\,\|u\|)\sigma'(0) + \lambda f(\lambda\,\|u\|)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\langle u, \epsilon\rangle\sigma'(t) + \langle u, p\times\epsilon\rangle p\times\epsilon)$$

$$= \cos(\lambda\,\|u\|)\epsilon + \langle u, \epsilon\rangle\lambda f(\lambda\,\|u\|)\sigma''(0) = \cos(\lambda\,\|u\|)\epsilon - \langle u, \epsilon\rangle\lambda f(\lambda\,\|u\|)p.$$

I wish to compute $\|u\|^2\partial_1\pi \odot \partial_2\pi(\eta(p))$. I know

$$\eta(p) = \frac{1}{\|u\|^2}u\wedge(p\times u), \qquad \text{whenever}\, u\neq 0.$$

So, I need to compute

$$\begin{aligned}\partial_{1,(u,p)}\pi(u) &= -\lambda f(\lambda\,\|u\|)(2\lambda'\,\|u\|^2+\lambda)\,\|u\|^2\,p + \lambda f(\lambda\,\|u\|)u\\ &\quad + (2\lambda'\cos(\lambda\,\|u\|) + \frac{\lambda}{\|u\|^2}f'(\lambda\,\|u\|))\,\|u\|^2\,u\\ &= -\lambda\,\|u\|^2\,f(\lambda\,\|u\|)(2\lambda'\,\|u\|^2+\lambda)p + (\lambda f(\lambda\,\|u\|)\\ &\quad + 2\lambda'\cos(\lambda\,\|u\|)\,\|u\|^2 + \lambda^2\,\|u\|\,f'(\lambda\,\|u\|))u\\ &= (2\lambda'\,\|u\|^2+\lambda)(\cos(\lambda\,\|u\|)u - \|u\|^2\,\lambda f(\lambda\,\|u\|)p),\end{aligned}$$

and

$$\partial_{1,(u,p)}\pi(p\times u) = \lambda f(\lambda\|u\|)p\times u,$$

and

$$\partial_{2,(u,p)}\pi(u) = \cos(\lambda\,\|u\|)u - \|u\|^2\,\lambda f(\lambda\,\|u\|)p$$

and finally

$$\partial_{2,(u,p)}\pi(p\times u) = \cos(\lambda\,\|u\|)p\times u.$$

Notice that

$$\partial_{1,(u,p)}\pi(u) = (2\lambda'\,\|u\|^2+\lambda)\partial_{2,(u,p)}\pi(u).$$

I can now compute

$$\begin{aligned}&\|u\|^2\,\partial_1\pi\odot\partial_2\pi(\eta(p))\\ &\quad = \partial_1\pi(u)\wedge\partial_2\pi(p\times u) + \partial_2\pi(u)\wedge\partial_2\pi(p\times u)\\ &\quad = \partial_2(u)\wedge((2\lambda'\,\|u\|^2+\lambda)\partial_{2,(u,p)}\pi(u) + \partial_1\pi(p\times u))\\ &\quad = (\cos(\lambda\,\|u\|)u - \|u\|^2\,\lambda f(\lambda\,\|u\|)p)\wedge((2\lambda'\,\|u\|^2+\lambda)\cos(\lambda\,\|u\|) + \lambda f(\lambda\,\|u\|))p\times u\\ &\quad = (\cos(\lambda\,\|u\|)u - \|u\|^2\sin(\lambda\,\|u\|)p)\wedge((2\lambda'\,\|u\|^2+\lambda)\cos(\lambda\,\|u\|) + \lambda f(\lambda\,\|u\|))p\times u.\end{aligned}$$

On the other hand

$$\|u\|^2\,\eta(\pi(u,p)) = \|u\|^2\,\phi_{\sigma(\lambda\|u\|)}(\eta(p)),$$

$$\text{with}\,\sigma(t) = \mathrm{Exp}\left(t\frac{u}{\|u\|}\right) = \cos(t)p + \sin(t)\frac{u}{\|u\|} = \phi_{\sigma(\lambda\|u\|)}(u)\wedge\phi_{\sigma(\lambda\|u\|)}(p\times u)$$

$$= \sigma'(\lambda\,\|u\|)\wedge(p\times u) = (\cos(\lambda\,\|u\|)u - \|u\|\sin(\lambda\,\|u\|)p)\wedge(p\times u).$$

It follows that π is a Poisson map if and only if λ satisfies the following differential equation:

$$(2\lambda'(t^2)t^2 + \lambda(t^2))\cos(\lambda(t^2)t) + \lambda(t^2) f(\lambda(t^2)t) = 1.$$

Put $\mu(t) = \lambda(t^2)$ so that $\mu'(t) = 2t\lambda'(t^2)$. The function μ satisfies the differential equation:

$$(\mu'(t)t + \mu(t))\cos(\mu(t)t) + \frac{\sin(\mu(t)t)}{t} = 1.$$

Put $\alpha(t) = \sin(\mu(t)t)$ so that $\alpha'(t) = (\mu'(t)t + \mu(t))\cos(\mu(t)t)$. The function α satisfies the differential equation:

$$t\alpha'(t) + \alpha(t) = t.$$

A general solution of this equation is

$$\alpha(t) = \frac{a}{t} + \frac{t}{2}, \quad \text{with}\, a \in \mathbb{R}.$$

Hence

$$\mu(t) = \frac{1}{t} \arcsin\left(\frac{a}{t} + \frac{t}{2}\right).$$

Since I want $\mu(0) = \lambda(0) = 1/2$, I need $a = 0$ and

$$\mu(t) = \frac{1}{t} \arcsin\left(\frac{t}{2}\right).$$

I deduce the following proposition.

Proposition A.1. *The map*

$$\pi : \mathrm{T}S^2 \to S^2, \qquad (u, p) \mapsto \mathrm{Exp}_p\left(\frac{1}{\|u\|}\arcsin\left(\frac{\|u\|}{2}\right)u\right)$$

is Poisson.

The way it is written in the previous proposition, the map π is only defined on a neighbourhood of S^2 in TS^2. Nevertheless, it can easily be extended to a continuous function on the whole of TS^2, so that it is Poisson wherever it is smooth. The fact that this map is not smooth at all points means that technics used in Section 6 will not carry over here. I nevertheless believe that a modification of these technics will make things work.

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